Exploring the Swift-Hohenberg Equation

HON292A: Entropy and Chaos: Order and Disorder in the Universe

11 September 2008

1 Pre-lab reading

Take a look at the images in *The Self-Made Tapestry* chapters entitled WAVES, BODIES, FLUIDS, COMMUNITIES and think about commonalities and categories you might use to classify them.
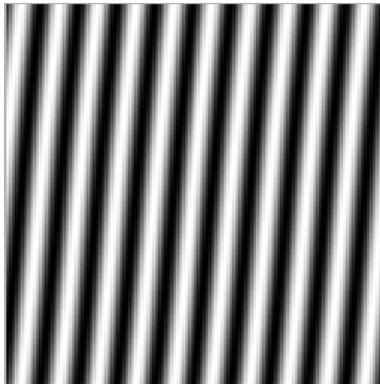
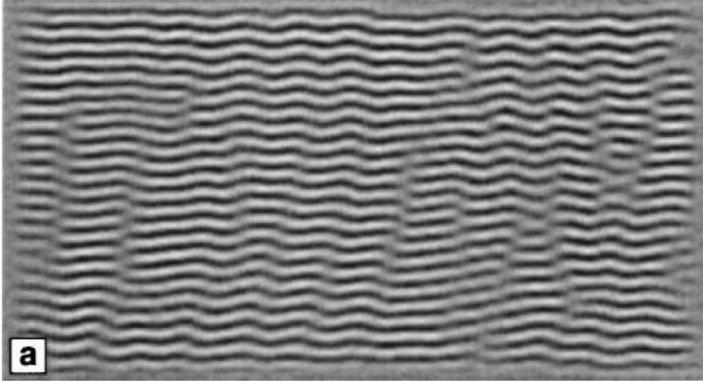## 2 The Swift-Hohenberg Equation

The Swift-Hohenberg equation

$$\frac{\partial \psi}{\partial t} = \epsilon \psi - (\nabla^2 + 1)^2 \psi + g\psi^2 - \psi^3$$

is a model pattern-forming equation which was derived by Jack Swift and Pierre Hohenberg as a model for a fluid which is thermally convecting. (The original paper is "Hydrodynamic fluctuations at the convective instability," *Physical Review A*, **15**: 319-328, 1977.) Since this equation likely has more advanced notation than you've seen before, we'll take it apart into its pieces.

You can think of the quantity $\psi(x, y)$ as representing a grayscale image of the temperature at each point in the $x-y$ plane. This means that each (x, y) coordinate has a temperature measurement (ψ) associated with that point. Therefore, an image of $\psi(x, y)$ might look like this

and represents a set of convection rolls where each white stripe is the hot fluid coming up, and each black stripe is the cold fluid going down. One reason this representation is used is that in experiments on thermal convection, scientists take pictures which look like this:

(from Daniels, Plapp, Bodenschatz, "Pattern Formation in Inclined Layer Convection," *Physical Review Letters*, **84:** 5320–5323, 2000)

Here, the black/white contrast is provided by a neat optical trick called *shadowgraphy*: light shining through the fluid layer is bent just a little bit by the fact that the rising fluid is less dense than the falling fluid.

The Swift-Hohenberg equation provides a solution for $\psi(x, y)$ by integrating forward in time and has the following features:

- The first thing to notice is that this equation relates the *temporal* evolution of the pattern to the *spatial* structure of the pattern. If you've had some calculus, note that $\frac{\partial}{\partial t}$ is a time derivative and ∇ is shorthand for a complicated spatial derivative: $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

- **Driving:** ϵ plays the role of a temperature knob, measuring how far the temperature is above the minimum temperature difference required for convection. Therefore, for $\epsilon < 0$, the heating at the bottom of the fluid is too small to cause convection, while for $\epsilon > 0$, convection occurs. Note that if the $\frac{\partial \psi}{\partial t} = \epsilon \psi$ were the whole equation, then we would just observe exponential growth (for $\epsilon > 0$) or decay (for $\epsilon < 0$).

- **Dissipation:** The term with the ∇ acts to smooth out sharp edges in pattern. (Note that the equation $\frac{\partial \psi}{\partial t} = -k \nabla^2 \psi$ is called a "diffusion equation.")

- **Feedback:** Both the ψ^2 and ψ^3 terms are *nonlinearities*, and g is the parameter controlling the strength of the quadratic nonlinearity. We'll discover what effect this has during our investigations below.

3   Download MATLAB Files

1. Return to the folder called `matlab` inside your K: drive ("My Documents") to make sure it's still there from last time.

2. Use your browser to go to `http://nile.physics.ncsu.edu/hon292a-f08/` (and you might want to bookmark it since we'll be going back here to download more files.)

3. Right-click on the files `sh_gui.m`, `sh_gui.fig`, and `swifthohen.m` and save them into your `matlab` folder.

4. Start MATLAB from the "Novell Applications" menu (under the heading Math & Science).

5. To run the programs, type `sh_gui` at the command line, and a window will open of which looks like this:
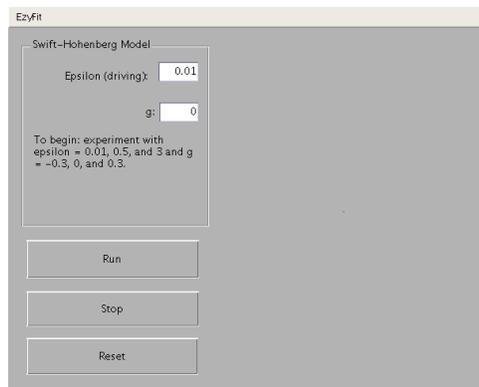
6. When you enter a value for ϵ and g in the boxes and hit $\boxed{\text{Run}}$ the computer will numerically integrate the Swift-Hohenberg Equation using those values, starting from random initial conditions. An image of the resulting convection rolls, as they change in time, will appear at the right side of the window.

   The $\boxed{\text{Stop}}$ button will end the simulation, and you can either use the $\boxed{\text{Reset}}$ button to start over completely or use the $\boxed{\text{Run}}$ button to start again using the same values of ϵ and g but with new random initial conditions.

4 Exploring the Swift-Hohenberg Equation

Perform the following numerical experiments, which we will use as fuel for upcoming writing and discussion topics. You might want to stick to a limited set of values for all experiments so that it's easier to compare: for instance $\epsilon = 0.01, 0.03, 0.1, 0.3, 1, 3...$.

In all cases, think about the following two questions: "What kind of symmetry-breaking occurs? What evidence of chaos do you see?"

1. For $\epsilon = 0.01$ and $g = 0$, run the simulation several times to get a feel for the range of behaviors possible at these single parameters. What do you observe?

2. For $\epsilon = 0.3$ and $g = 0$, run the simulation several times to get a feel for the range of behaviors possible at these single parameters. What do you observe?

3. Keeping $\epsilon = 0.01$, what effect does g have on the final state of the system?

4. Keeping $g = 0$, what effect does ϵ have on the final state of the system?

5. Keeping $g = 0.3$, what effect does ϵ have on the final state of the system?

6. Is there any difference between $\pm g$?